**Table 1 Nondimensional radiosity at the first and second gaussian quadrature points for the case of infinite adjoint plates**

| $\rho$ | $\theta$ | $X = x/L$ | $B(x)$ $n = 3$ | $B(x)$ $n = 15$ | $Z$ | $B_r(x)$ | $E$, % |
|---|---|---|---|---|---|---|---|
| 0.1 | 90 | 0.1127 | 1.0576 | | 0.9565 | 1.0463 | 1.07 |
| | | 0.5000 | 1.0282 | | 0.9843 | 1.0285 | 0 |
| | | 0.0060 | | 1.0683 | 0.9548 | 1.0522 | 1.14 |
| | | 0.0314 | | 1.0506 | 0.9802 | 1.0508 | 0 |
| | 60 | 0.1127 | 1.1155 | | 0.9077 | 1.0753 | 3.74 |
| | | 0.5000 | 1.0524 | | 0.9667 | 1.0526 | 0 |
| | | 0.0060 | | 1.1237 | 0.9042 | 1.0807 | 3.97 |
| | | 0.0314 | | 1.0814 | 0.9580 | 1.0795 | 0 |
| | 45 | 0.1127 | 1.1708 | | 0.8628 | 1.0850 | 8.84 |
| | | 0.5000 | 1.0711 | | 0.9491 | 1.0683 | 0 |
| | | 0.0060 | | 1.1807 | 0.8575 | 1.0931 | 8.00 |
| | | 0.0314 | | 1.1006 | 0.9375 | 1.0920 | 0 |
| 0.5 | 90 | 0.1127 | 1.3613 | | 0.7825 | 1.2757 | 6.72 |
| | | 0.5000 | 1.1629 | | 0.9215 | 1.1620 | 0 |
| | | 0.0060 | | 1.4218 | 0.7742 | 1.3262 | 7.20 |
| | | 0.0314 | | 1.3129 | 0.9010 | 1.3129 | 0 |
| | 60 | 0.1127 | 2.0040 | | 0.5387 | 1.5205 | 31.90 |
| | | 0.5000 | 1.3479 | | 0.8335 | 1.3335 | 1.08 |
| | | 0.0060 | | 2.1457 | 0.5210 | 1.5868 | 35.20 |
| | | 0.0314 | | 1.6015 | 0.7900 | 1.5706 | 1.90 |
| | 45 | 0.1127 | 3.4204 | | 0.3144 | 1.6725 | 104.00 |
| | | 0.5000 | 1.5429 | | 0.7456 | 1.4644 | 5.10 |
| | | 0.0060 | | 3.8447 | 0.2879 | 1.7318 | 122.00 |
| | | 0.0314 | | 1.8626 | 0.6878 | 1.7184 | 8.40 |
| 0.9 | 90 | 0.1127 | 1.8659 | | 0.6086 | 1.6131 | 15.70 |
| | | 0.5000 | 1.3521 | | 0.8587 | 1.3403 | 0.88 |
| | | 0.0060 | | 2.0818 | 0.5935 | 1.7718 | 17.50 |
| | | 0.0314 | | 1.7326 | 0.8218 | 1.7265 | 0.35 |
| | 60 | 0.1127 | 7.2931 | | 0.1697 | 2.4867 | >200.00 |
| | | 0.5000 | 2.2249 | | 0.7003 | 1.8364 | 20.60 |
| | | 0.0060 | | 10.3946 | 0.1378 | 2.8729 | >200.00 |
| | | 0.0314 | | 3.3463 | 0.6220 | 2.7554 | 21.40 |

$Z$ approaches one the error between the approximate solutions and the convergent solutions approaches zero.

Application of this analysis to problems of two variables follows immediately. That is, for a gray enclosure

$$B(x,y) = 1 + \rho \int\int B(\xi,\eta) K(x,y,\xi,\eta) d\xi d\eta$$

which may be approximated by

$$B(x_i,x_j) = 1 + \rho \sum_{1} \sum_{k} B(x_1,x_k) K(x_i,x_j,x_1,x_k) W_1 W_k$$

Rearrangement and application of the same line of reasoning yields

$$Z_2 = 1 - \rho W_i W_j K(x_i,x_j,x_i,x_j) > 0 \quad (7)$$

Application of various quadrature formulas to Eq. (7) for the case of finite parallel and adjoint plates indicates that the parallel plates case may be easily solved by any quadrature method, but that not even gaussian quadrature may be applied to the adjoint plates case with any reliability.

If the parameter $Z$ is less than zero, Eqs. (3) or its equivalent for the two variable case are impossible from a physical standpoint. Thus, an apparent singular point of the integral is not adequately approximated by numerical quadrature. The results of this study seems to imply that the numerical quadrature approximation tends to over estimate the kernel of the integral equation for points directly opposed which are near the apparent singularity of the integral.

Thus, parameters have been introduced which yield information essential to the use of numerical quadrature as a method of solution of integral equations encountered in radiative heat-transfer computations. These parameters should approach one as the order of quadrature or the number of intervals is increased before valid approximate solutions may be obtained.

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# Maximum Errors from Quantization in Multirate Digital Control Systems

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A PROBLEM in the implementation of digital control systems is the determination of the number of bits required to adequately represent signals within the digital controller. The quantization of signals within the controller introduces errors into the system output, and the magnitudes of the errors are a direct function of the number of bits used to represent signals within the controller. Techniques have been developed which yield both the mean-square error and an upper bound on the errors for single rate systems,[1–3] and that yield the mean-square error for a multirate system.[4] This paper presents a technique for determining an upper bound on the system errors due to quantization in a multirate controller.

### Multirate System

Consider the multirate control system shown in Fig. 1. The $z$-transform variables are defined as $z = \epsilon^{sT}$, and $z_n = z^{1/n}$. For the system

$$Y(z) = \{\mathfrak{z}[D(z_n)G(z_n)]R(z)\}/\{1 + \mathfrak{z}[D(z_n)G(z_n)]\} \quad (1)$$

In Eq. (1),

$$G(z_n) = \mathfrak{z}n\{(1 + \epsilon^{-Ts/n})\, G(s)/s\} \quad (2)$$

i.e., the pulse transfer function of the data-hold and plant combination, with sampling period $T/n$. For notation, let

$$G(z_n)|_{z_n = z^{1/n}} = G(z)_n$$

Now let

$$G_f(z)_n = D(z)_n G(z)_n = g_f(0) + g_f(1/n)z^{-1/n} + \ldots + g_f(1)z^{-1} + \ldots \quad (3)$$

Then the $z$-transform of $D(z)_n G(z)_n$ is

$$\mathfrak{z}[D(z)_n G(z)_n] = \mathfrak{z}[D(z_n)G(z_n)] = g_f(0) + g_f(1)z^{-1} + g_f(2)z^{-2} + \ldots \quad (4)$$

It is desired now to investigate the errors that appear in the output due to quantization within the digital controller. The digital controller transfer function, $D(z_n)$, is realized by some

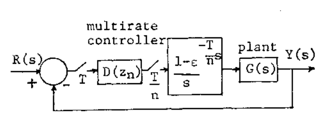


**Fig. 1 Multirate control system.**

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programing form.[5] For example, suppose that $D(z_n)$ is the second-order function

$$D(z_n) = (z_n^2 + \alpha_1 z_n + \alpha_0)/(z_n^2 + \beta_1 z_n + \beta_0) \quad (5)$$

The canonical programing form for the realization of this transfer function is shown in Fig. 2. Quantization occurs in each of the blocks labeled $Q$. The quantization process can be represented by an additive noise source, as shown in Fig. 2 (Ref. 1). For example, the noise source $Q_2$ has a $z$-transform

$$Q_2(z)_n = q_2(0) + q_2(1/n)z^{-1/n} + q_2(2/n)z^{-2/n} + \ldots \quad (6)$$

where $q_2(i/n)$ is the error introduced by the second quantizer at time $t = iT/n$. Let $h$ be the granularity of the quantizer. Then $q_2(i/n) \leq h$ for truncation, and $q_2(i/n) \leq h/2$ for roundoff.

Consider now a noise source $Q$ within the digital controller. The system output will be derived as a function of $Q(z_n)$, with $R(z) = 0$. This output is then the error caused by the quantizer. Let $G_p(z_n)$ be the transfer function from this quantizer noise source to the output $Y(z_n)$, with the system opened at the slow-rate sampler. Then the output caused by the quantizer noise source $Q(z_n)$ is

$$Y(z_n) = G_p(z_n)Q(z_n) - D(z_n)G(z_n)Y(z) \quad (7)$$

Taking the $z$-transform of Eq. (7) yields

$$Y(z) = \mathfrak{Z}[Y(z_n)] = \mathfrak{Z}[G_p(z_n)Q(z_n)] - \mathfrak{Z}[D(z_n)G(z_n)]Y(z) \quad (8)$$

Then Eq. (8) may be solved for $Y(z)$

$$Y(z) = \mathfrak{Z}[G_p(z_n)Q(z_n)]/\{1 + \mathfrak{Z}[D(z_n)G(z_n)]\} \quad (9)$$

It is desired to determine an upper bound on $y(k)$ in Eq. (9), given that $q(i/n)$ is bounded by the quantizer granularity $h$, as stated previously.

Now

$$G_p(z_n) = g_p(0) + g_p(1/n)z^{-1/n} + g_p(2/n)z^{-2/n} + \ldots \quad (10)$$

and

$$Q(z)_n = q(0) + q(1/n)z^{-1/n} + q(2/n)z^{-2/n} + \ldots \quad (11)$$

For $Q(z)_n = q(i/n)z^{-i/n}$

$$Y_i(z) = \mathfrak{Z}[G_p(z_n)z^{-i/n}]q(i/n)/\{1 + \mathfrak{Z}[D(z_n)G(z_n)]\} \quad (12)$$

Suppose, in Eq. (12), that $i = kn + j$, where $j$ and $k$ are integers. Then

$$Y_i(z) = [G_p(z_n)z^{-j/n}]q(i/n)z^{-k}\}/\{1 + \mathfrak{Z}[D(z_n)G(z_n)]\} \quad (13)$$

Then the total output given in Eq. (9) may be expressed as

$$Y(z) = \frac{\mathfrak{Z}[G_p(z_n)]}{1 + \mathfrak{Z}[D(z_n)G(z_n)]}[q(0) + q(1)z^{-1} + q(2)z^{-2} + \ldots] \frac{\mathfrak{Z}[G_p(z_n)z^{-1/n}]}{1 + \mathfrak{Z}[D(z_n)G(z_n)]}\left[q\left(\frac{1}{n}\right) + q\left(n + \frac{1}{n}\right)z^{-1} + q\left(2n + \frac{1}{n}\right)z^{-2} + \ldots\right] + \ldots + \frac{\mathfrak{Z}[G_p(z_n)z^{-(n-1)/n}]}{1 + \mathfrak{Z}[D(z_n)G(z_n)]}\left[q\left(n - \frac{1}{n}\right) + q\left(2n - \frac{1}{n}\right)z^{-1} + q\left(3n - \frac{1}{n}\right)z^{-3} + \ldots\right] \quad (14)$$

Now, let

$$\frac{\mathfrak{Z}[G_p(z_n)z^{-i/n}]}{1 + \mathfrak{Z}[D(z_n)G(z_n)]} = g_i\left(-\frac{i}{n}\right) + g_i\left[\left(\frac{n-i}{n}\right)\right]z^{-1} + g_i\left[\left(\frac{2n-i}{n}\right)\right]z^{-2} + \ldots \quad (15)$$

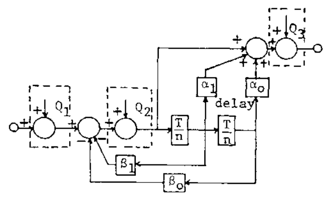

**Fig. 2 Second-order programing form showing quantization.**

Regrouping the terms in Eq. (14), we obtain the signal $y(t)$ at $t = kT$ as

$$y(k) = g_0(k)q(0) + g_1[(kn-1)/n]q(1/n) + \ldots + g_{n-1}\{[n(k-1)+1]/n\}\,q[(n-1)/n] + g_0(k-1)q(1) + \ldots + g_{n-1}(1/n)q[(kn-1)/n] + g_0(0)q(k) \quad (16)$$

Then, in Eq. (16), $y(k)$ is seen to be maximum if the $q_i$ are given by

$$\begin{aligned} q(0) &= h\,\mathrm{sgn}[g_0(k)] \\ q(1/n) &= h\,\mathrm{sgn}\{g_1[(kn-1)/n]\} \\ &\;\vdots \\ q(k) &= h\,\mathrm{sgn}[g_0(0)] \end{aligned} \quad (17)$$

assuming that the quantizer truncates. The maximum possible value of $y(k)$ is given by

$$y(k)_{\max} = h\left[\sum_{j=0}^{k}|g_0(j)| + \sum_{j=0}^{k}\left|g_1\left[\frac{(jn+1)}{n}\right]\right| + \sum_{j=0}^{k}\left|g_2\left[\frac{(jn+2)}{n}\right]\right| + \sum_{k=0}^{k}\left|g_{n-1}\left\{\frac{[n(j+1)-1]}{n}\right\}\right|\right] \quad (18)$$

Thus Eq. (18) is the desired result, yielding the maximum possible error in the system output at any sampling instant $t = kT$.

### Evaluation of Maximum Error

Equation (18) gives an upper bound on the quantization error in the system output at any sampling instant $t = kT$. In general, it will be quite difficult to evaluate the values $g_i$ in the above expression by a $z$-transform approach. It is seen that the sequence $[g_0(k)]$ is the system impulse response. This sequence may be obtained from a digital simulation of the system using the input sequence $q(0) = 1$ and $q(l/n) = 0$, $l = 1, 2, \ldots$. The impulse response sequence $[g_i(k)]$ may be obtained from a digital simulation of the system, using the input sequence $q(i/n) = 1$, $i < n$, and $q(l/n) = 0$, $l = 0, 1, 2, \ldots, l \neq i$. This technique has been applied to a model of the Saturn V attitude control system which had a 27th order plant. The fast-rate sampling frequency was 25 Hz, and the slow-rate sampling frequency was 2.5 Hz. The multirate compensator was third-order, preceded by a digital zero-order hold. Approximate twenty-nine minutes of computer time on the IBM 360/50 model was required to evaluate Eq. (18) for $k = 2000$, for one quantizer.

### Conclusions

A technique is presented for determining an upper bound on system errors caused by quantization in a multirate system. Even though the development is in terms of the $z$-transform, it is recommended that a state-variable approach be used to evaluate the upper bound. It is felt that the state-variable approach is simpler than attempting to find the required $z$-transform transfer functions. Also, since normally a digital simulation is required to evaluate the design of a system, the required state-variable formulation will already be available.

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# Stability of a Dual-Spin Satellite with a Four-Mass Nutation Damper


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THE purpose of this Note is to suggest a four-mass nutation damper for use in dual-spin satellites. It is shown that the stability criteria developed for this damper do not involve the kind of design constraints which restricted the performance of the other available nutation dampers.[1–3] The result is found to hold even when a circular disk or wheel of uniform mass distribution is used to replace the four-mass structure in the proposed design.

### Dual-Spin System

Consider the dual-spin configuration illustrated in Fig. 1. The main body of the satellite is a right circular cylinder with the nominal spin axis in the $a_3$ direction and a small rotor or fly-wheel is attached to it whose spin axis is also assumed to be along the $a_3$ body axis. The nutation damper placed only on the main body of the satellite, is a four-mass structure constrained to move in the $a_2$-$a_3$ plane as shown. Let 0 represent the center of mass of the whole system, and since the damper masses are arranged as collinear pairs, it can be assumed that there will be no appreciable shift in the center of mass of the system as the damper masses execute motions. It is also assumed that the four-mass structure is pivoted on a torsional system which offers dissipative torque in addition to the restoring torque. Although the results derived here will be found valid when a relatively more attractive structure—a wheel of uniform mass distribution—is used to replace the four-mass design shown in Fig. 1, the present analysis will be carried out for this latter design, since the four-mass configuration may be shown to be the simplest elementary structure which yields the desired results.

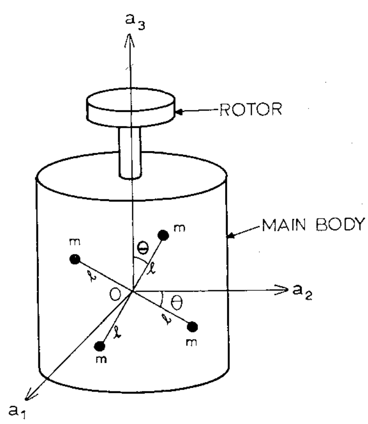


**Fig. 1 The dual-spin system.**

### Equations of Motion

For the system of Fig. 1, the equation of motion in a torque-free environment may be written as[2]

$$\pi'\cdot\boldsymbol{\omega} + \pi\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\pi\cdot\boldsymbol{\omega}) + \dot{\mathbf{h}} + m\sum_{i=1}^{4}\mathbf{r}_i\times\mathbf{r}''_i + m\sum_{i=1}^{4}\boldsymbol{\omega}\times(\mathbf{r}_i\times\mathbf{r}'_i) = 0 \quad (1)$$

where $\pi$ represents the inertia dyadic of the total system about the point 0, $\boldsymbol{\omega}$ is the angular velocity of the main body of the satellite in inertial space, $\mathbf{r}_i$ is the vector from 0 to the damper masses $m$, and $\mathbf{h}$ is the relative angular momentum of the rotor and where the prime over a quantity indicates the time derivative in the reference body frame, while the dot indicates inertial differentiation.

First, in evaluating the terms in Eq. (1) involving the inertia dyadics, note that the effective moments and products of inertia for the system of Fig. 1 are given by

$$I_{11} = I_1, I_{22} = I_2, I_{33} = I_3$$

and

$$I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32} = 0 \quad (2)$$

where $I_1$, $I_2$, and $I_3$ represent the moments of inertia of the dual-spin system under consideration when the damper masses are not deflected from their initial positions on the body axes.

Also, by definition,

$$\pi = \Sigma I_{\alpha\beta}a_\alpha a_\beta \quad (3)$$

Therefore, by using Eqs. (2) and (3), the terms involving the inertia dyadics in Eq. (1) may be evaluated as

$$\pi'\cdot\boldsymbol{\omega} + \pi\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\pi\cdot\boldsymbol{\omega}) = a_1[I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)] + a_2[I_2\dot{\omega}_2 - \omega_1\omega_3(I_3 - I_1)] + a_3[I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2)] \quad (4)$$

Next, the term $\dot{\mathbf{h}}$ which represents the contribution from the rotor to the basic equation of motion, may be found in the component form as

$$\dot{\mathbf{h}} = a_1I_R\omega_2\Omega - a_2I_R\omega_1\Omega + a_3I_R\dot{\Omega} \quad (5)$$

where $I_R$ denotes the spin-axis moment of inertia of the rotor and $\Omega$ is the angular speed of the rotor relative to the main body of the satellite.

Finally, by expressing the vector quantity $\mathbf{r}_i$ in terms of its $a_2$ and $a_3$ components, it can be shown that the evaluation of the two summation terms in Eq. (1) yields

$$m\sum_{i=1}^{4}\mathbf{r}_i\times\mathbf{r}''_i + m\sum_{i=1}^{4}\boldsymbol{\omega}\times(\mathbf{r}_i\times\mathbf{r}'_i) = -a_1I_m\ddot{\theta} - a_2I_m\omega_3\dot{\theta} + a_3I_m\omega_2\dot{\theta} \quad (6)$$

where $I_m = 4ml^2$ is the moment of inertia of the damper mass structure.

Now, by combining Eqs. (4, 5, and 6), one obtains, under the assumed torque-free condition, the following set of non-

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